

Neural ODE core

$$\dot{h}(t) = f_\theta(t, h(t)), \quad h(t_0) = h_0$$

Here f_θ is a tiny hard-coded “neural” vector field:

$$f_\theta(t, h) = 0.9 \sigma(1.2h + 0.7t - 0.3) + 0.1, \quad \sigma(z) = \frac{z}{1 + |z|}$$

Runge–Kutta (RK) one-step map

An explicit RK method with tableau (A, b, c) defines a discrete flow:

$$\Phi_{(A,b,c)}^{\Delta t} : h_n \mapsto h_{n+1}$$

via stages

$$k_i = f_\theta(t_n + c_i \Delta t, h_n + \Delta t \sum_{j < i} a_{ij} k_j), \quad h_{n+1} = h_n + \Delta t \sum_i b_i k_i.$$

Stage permutation (Theorem 382A ingredient)

A permutation p of the stages produces a new tableau $(A^{(p)}, b^{(p)}, c^{(p)})$ by consistently reindexing **all** coefficients:

$$b_i^{(p)} = b_{p(i)}, \quad c_i^{(p)} = c_{p(i)}, \quad a_{ij}^{(p)} = a_{p(i), p(j)} \quad (i > j),$$

which yields an **equivalent RK one-step map**:

$$\Phi_{(A,b,c)}^{\Delta t} \equiv \Phi_{(A^{(p)}, b^{(p)}, c^{(p)})}^{\Delta t}.$$

Composition tested in the code

The code composes two RK solvers (RK4 then Heun) as a “macro-step”:

$$h_A = \Phi_{\text{Heun}}^{\Delta t}(t_0 + \Delta t, \Phi_{\text{RK4}}^{\Delta t}(t_0, h_0))$$

and compares it with the composition of their permuted-but-equivalent forms:

$$h_B = \Phi_{\text{Heun}^{(p_1)}}^{\Delta t}(t_0 + \Delta t, \Phi_{\text{RK4}^{(p_2)}}^{\Delta t}(t_0, h_0)).$$

What the printout demonstrates (Theorem 382A)

Because each permuted tableau defines the **same one-step map**, the composed result is unchanged:

$$h_A = h_B \quad (\text{up to floating-point roundoff}),$$

so the reported $|h_A - h_B|$ should be near machine precision.